e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Title/Heading of topic:Tests for convergence of infinite series (Raabe's test, logarithmic test, Integral test)

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∴. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

(iii) Test fails if $l = 1$

Example Test the convergence of the following series:

(i) $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \ldots..$ (ii) $1 + \frac{3x}{7} + \frac{3.6\,x^2}{7.10} + \frac{3.6.9x^3}{7.10.13} + \cdots (x > 0)$

Solution:(i) Here $u_n = \frac{2.4.6....2n}{1.3.5....(2n+1)} \Rightarrow u_{n+1} = \frac{2.4.6....2n(2n+2)}{1.3.5....(2n+1)(2n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+2}{2n+3} = 1$

Hence Ratio test fails.

Now applying Raabe's test, we have

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+3}{2n+2} - 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

Hence by Raabe's test , the given series diverges.

(ii) Ignoring the first term, $u_n = \frac{3.6.9...3n}{7.10.13...(3n+4)} x^n$

$$\Rightarrow u_{n+1} = \frac{3.6.9...3n(3n+3)}{7.10.13...(3n+4)(3n+7)} x^{n+1}$$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+7} x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim_{n\to\infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

Hence by Raabe's test, the given series converges if $x = 1$

∴ the given series converges if $x \leq 1$ and diverges if $x > 1$.

. Logarithmic Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n \frac{u_n}{u_{n+1}} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

Example. Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$$

Solution: Here $u_n = \frac{n^n x^n}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\text{Then } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}x}{(n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n x}{n^n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n x = e.x$$

Hence by Ratio test , the given series converges if $ex < 1$ $i.e. x < \frac{1}{e}$ and diverges if $ex > 1$ $i.e. x > \frac{1}{e}$

Test fails if $ex = 1$ $i.e. x = \frac{1}{e}$

Since $\frac{u_{n+1}}{u_n}$ involves e ∴ applying logarithmic test.

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n x}$$

∴ for $x = \frac{1}{e}$, $\frac{u_n}{u_{n+1}} = e.\frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$log\left(\frac{u_n}{u_{n+1}}\right) = log\, e - log\left(1+\frac{1}{n}\right)^n = 1 - nlog\left(1+\frac{1}{n}\right)$$

$$=1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right)$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

$$= \lim_{n\to\infty} nlog\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\left(\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right) = \frac{1}{2} < 1$$

∴ By logarithmic test , the series diverges for $x = \frac{1}{e}$.

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

.Cauchy's Integral Test

If $u(x)$ is non-negative , integrable and monotonically decreasing function such that $u(n)=u_n$, then if $\int_1^\infty u(x)\, d(x)$ converges then the series $\sum_{n=1}^{\infty} u_n$ also converges.

Example Test the convergence of the following series

$(i)\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ $(ii)\ \sum_{n=2}^{\infty}\frac{1}{n(\log n)}$

Solution:(i) Here $u_n = \frac{1}{n^2+1}$.

Let $u(x) = \frac{1}{x^2+1}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_1^{\infty}\frac{1}{x^2+1}\,d(x) = [tan^{-1}x]_1^{\infty}$

$= tan^{-1}\infty - tan^{-1}1$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ which is finite.

Hence $\int_1^{\infty}\frac{1}{x^2+1}\,d(x)$ converges so $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ also converges.

(ii) Here $u_n = \frac{1}{n(\log n)}$.

Let $u(x) = \frac{1}{x(\log x)}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_2^{\infty}\frac{1}{x(\log x)}\,d(x) = \log(log\infty) - \log\,(log2) = \infty$

Hence $\sum_{n=2}^{\infty}\frac{1}{n(\log n)}$ diverges.